arXiv:1504.02172v1 [math.AP] 9 Apr 2015

Well-posedness for Stochastic Generalized Fractional Benjamin-Ono Equation *

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Abstract. This paper is devoted to the Cauchy problem for the stochastic generalized Benjamin-Ono equation. By using the Bourgain spaces and Fourier restriction method and the assumption that u_0 is $\mathcal{F}_0$-measurable, we prove that the Cauchy problem for the stochastic generalized Benjamin-Ono equation is locally well-posed for the initial data $u_0(x,w) \in L^2(\Omega; H^s(\mathbf{R}))$ with $s \geq \frac{1}{2} - \frac{\alpha}{4}$, where $0 < \alpha \leq 1$. In particular, when $u_0 \in L^2(\Omega; H^{\frac{\alpha+1}{2}}(\mathbf{R})) \cap L^{\frac{2(2+3\alpha)}{\alpha}}(\Omega; L^2(\mathbf{R}))$, we prove that there exists a unique global solution $u \in L^2(\Omega; H^{\frac{\alpha+1}{2}}(\mathbf{R}))$ with $0 < \alpha \leq 1$.

Keywords: Cauchy problem; Stochastic fractional Benjamin-Ono equation, bilinear estimate

Mathematics Subject Classification(2000): 35E15, 35Q53, 60H15

*Supported by the NSF of China (No.11371367) and Fundamental program of NUDT(JC12-02-03)

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1 Introduction

In this paper, we consider the following stochastic generalized fractional Benjamin-Ono type equation

$$\begin{cases} du(t) = [-|D_x|^{\alpha+1}\partial_x u(t) + u(t)^k u_x(t)]dt + \Phi dW(t), \\ u(0) = u_0 \end{cases} \tag{1.1}$$

where $0 < \alpha \leq 1$ and $|D_x|$ is the Fourier multiplier operator with symbol $|\xi|$. We recall that the Benjamin-Ono equation is a nonlinear partial integro-differential equation that describes one-dimensional internal waves in deep water, which was introduced by Benjamin [1] and Ono[34].

In fact, equation (1.1) is equivalent to the following equations:

$$\begin{cases} \frac{du(t)}{dt} = [-|D_x|^{1+\alpha}\partial_x u(t) + u(t)^k u_x(t)] + \Phi\frac{dW(t)}{dt}, \\ u(0) = u_0. \end{cases} \tag{1.2}$$

Equation (1.2) is considered as the Benjamin-Ono type equation

$$\begin{cases} \frac{dv(t)}{dt} = [-|D_x|^{1+\alpha}\partial_x v(t) + v^k(t)v_x(t)], \\ u(0) = u_0. \end{cases} \tag{1.3}$$

forced by a random term. (1.3) which contains Benjamin-Ono equation and KdV equation arise as mathematical models for the weakly nonlinear propagation of long waves in shallow channels.

When $\alpha = 1$ and $k = 1$, (1.1) reduces to the stochastic KdV equation which has been studied by some people, we refer the readers to [2, 3, 4].

When $\alpha = 1$ and $k = 1$, (1.3) reduces to the KdV equation which has been investigated by many authors, we refer the readers to [5, 20, 21, 7, 8, 9, 10, 14, 15, 22, 23, 24, 26]. The result of [23] and [24] implies that $s = -\frac{3}{4}$ is the critical well-posedness indices for the Cauchy problem for the KdV equation. Guo[15] and [26] almost proved that the KdV equation is globally well-posed in $H^{-3/4}$ with the aid of I-method and the dyadic bilinear estimates at the same time. When $\alpha = 1$ and $k = 2$, (1.3) reduces to the mKdV equation which has been studied by some people, we refer the readers to [10, 15, 22, 26, 35]. Recently, Chen et.al [6] studied the Cauchy problem for the stochastic Camassa-Holm equation. When $\alpha = 0$ and $k = 1$, (1.3) reduces to the Benjamin-Ono equation which has been studied by many people, we refer the readers to [27, 28, 29, 30, 31, 32, 33, 37]. By using the gauge transformation introduced by [37] and a new bilinear estimate, Ionescu and Kenig [25] proved that the Benjamin-Ono equation is globally well-posed in $H^s(\mathbf{R})$ with $s \geq 0$. When $0 < \alpha \leq 1$ and $k = 1$, (1.3) has been investigated by some people, we refer the readers to [11, 13, 17, 18, 20]. In [18], the author proved that (1.3) is locally well-posed in $H^{(s,w)}$ and globally well-posed in $H^{(0,w)}$. Recently, by using a frequency dependent renormalization method, Herr [19] proved that (1.3) is globally well-posed in L^2 if $0 < \alpha < 1$. Very recently, Guo [16] proved that (1.3) is globally well-posed in H^s with $s \geq 1-\alpha$ if $0 \leq \alpha \leq 1$

with $k=1$ and in H^s with $s\geq \frac{1}{2}-\frac{\alpha}{4}$. Richards prove a global well-posedness result for stochastic KDV-Burgers equation under an additional smoothing of the noise, we refer to [36] for the details.

In this paper, we focus the case $0<\alpha\leq 1$ and $k=2$ of (1.1). In this paper, we consider the Cauchy problem for the stochastic generalized Benjamin-Ono equation. By using the Bourgain spaces and Sobolev spaces and the assumption that u_0 is $\mathcal{F}_0$- measurable and $\Phi\in L_2^{0,\frac{\alpha+1}{2}}$., we prove that the Cauchy problem for the stochastic generalized Benjamin-Ono equation is locally well-posed for the initial data $u_0(x,w)\in L^2(\Omega; H^s(\mathbf{R}))$ with $s\geq\frac{1}{2}-\frac{\alpha}{4}$, where $0<\alpha\leq 1$. In particular, when $u_0\in L^2(\Omega; H^{\frac{\alpha+1}{2}}(\mathbf{R}))\cap L^{\frac{2(2+3\alpha)}{\alpha}}(\Omega; L^2(\mathbf{R}))$ and $\Phi\in L_2^{0,\frac{\alpha+1}{2}}$, we prove that there exists a unique global solution $u\in L^2(\Omega;H^{\frac{\alpha+1}{2}}(\mathbf{R}))$ if $0<\alpha\leq 1$.

We give some notations before giving the main result. We denote $X\sim Y$ by $A_1|X|\leq |Y|\leq A_2|X|$, where $A_j>0 (j=1,2)$ and denote $X\gg Y$ by $|X|>C|Y|$, where C is some positive number which is larger than 2. $\langle\xi\rangle^s=(1+\xi^2)^{\frac{s}{2}}$ for any $\xi\in\mathbf{R}$, and $\mathscr{F}u$ denotes the Fourier transformation of u with respect to its all variables. $\mathscr{F}^{-1}u$ denotes the Fourier inverse transformation of u with respect to its all variables. $\mathscr{F}_x u$ denotes the Fourier transformation of u with respect to its space variable. $\mathscr{F}_x^{-1}u$ denotes the Fourier inverse transformation of u with respect to its space variable. is the Schwartz space and is its dual space. $H^s(\mathbf{R})$ is the Sobolev space with norm $\|f\|_{H^s(\mathbf{R})}\|\langle\xi\rangle^s\mathscr{F}_x f\|_{L^2_\xi(\mathbf{R})}$. For any $s,b\in\mathbf{R}$, $X_{s,\,b}(\mathbf{R}^2)$ is the Bourgain space with phase function $\phi(\xi)=\xi|\xi|^{1+\alpha}$. That is, a functions $u(x,t)$ in belongs to $X_{s,b}(\mathbf{R}^2)$ iff

$$\|u\|_{X_{s,\,b}(\mathbf{R}^2)}=\left\|\langle\xi\rangle^s\langle\tau-\xi|\xi|^{\alpha+1}\rangle^b\mathscr{F}u(\xi,\tau)\right\|_{L^2_\tau(\mathbf{R})L^2_\xi(\mathbf{R})}<\infty.$$

For any given interval L, $X_{s,\,b}(\mathbf{R}\times L)$ is the space of the restriction of all functions in $X_{s,\,b}(\mathbf{R}^2)$ on $\mathbf{R}\times L$, and for $u\in X_{s,\,b}(\mathbf{R}\times L)$ its norm is

$$\|u\|_{X_{s,\,b}(\mathbf{R}\times L)}=\inf\{\|U\|_{X_{s,\,b}(\mathbf{R}^2)}; U|_{\mathbf{R}\times L}=u\}.$$

When $L=[0,T]$, $X_{s,\,b}(\mathbf{R}\times L)$ is abbreviated as $X_{s,b}^T$. Throughout this paper, we always assume that $w(\xi)=\xi|\xi|^{\alpha+1}$, ψ is a smooth function, $\psi_\delta(t)=\psi(\frac{t}{\delta})$, satisfying $0\leq\psi\leq 1$, $\psi=1$ when $t\in[0,1]$, supp$\psi\subset[-1,2]$ and $\sigma=\tau-\xi|\xi|^{\alpha+1}$, $\sigma_k=\tau_k-\xi_k|\xi_k|^{\alpha+1}$ $(k=1,2)$,

$$\begin{aligned}
U(t)u_0 &= \int_{\mathbf{R}} e^{i(x\xi-\xi|\xi|^{\alpha+1})}\mathscr{F}_x u_0(\xi)d\xi,\\
\|f\|_{L^q_tL^p_x} &= \left(\int_{\mathbf{R}}\left(\int_{\mathbf{R}}|f(x,t)|^p dx\right)^{\frac{q}{p}}dt\right)^{\frac{1}{q}},\\
\|f\|_{L^p_tL^p_x} &= \|f\|_{L^p_{xt}}.
\end{aligned}$$

We assume that $B(x,t)$, $t\geq 0, x\in\mathbf{R}$, is a zero mean gaussian process whose covariance function is given by

$$\mathbf{E}(B(t,x)B(s,y))=(t\wedge s)(x\wedge y)$$

for $t, s \geq 0, x, y \in \mathbf{R}$ and $W(t) = \frac{\partial B}{\partial x} = \sum_{i=1}^{\infty} \beta_j e_j$, where $(e_i)_{i\in \mathrm{N}}$ is an orthonormal basis of $L^2(\mathbf{R})$ and (β_i) is a sequence of mutually independent real brownian motions in a fixed probability space, is a cylindrical Wiener process on $L^2(\mathbf{R})$. $(.,.)$ denotes the L^2 space duality product, i.e., $(f, g) = \int_{\mathbf{R}} f(x)g(x)dx$. $(\Omega, \mathscr{F}, \mathbf{P})$ is a probability space endowed with a filtration $(\mathscr{F}_t)_{t\geq 0}$. $\mathbf{E}f = \int_\Omega f d\mathbf{P}$. $W(t)$ is a cylindrical Wiener process $(W(t))_{t\geq 0}$ on $L^2(\mathbf{R})$ associated with the filtration $(\mathscr{F})_{t\geq 0}$. For any orthonormal basis $(e_k)_{k\in\mathbf{N}}$ of $L^2(\mathbf{R})$, $W = \sum_{k=0}^{\infty} \beta_k e_k$ for a sequence $(\beta_k)_{k\in\mathbf{N}}$ of real, mutually independent brownian motions on $(\Omega, \mathscr{F}, \mathbf{P}, \mathscr{F}_t)_{t\geq 0})$. Let H be a Hilbert space, $L_2^0(L^2(\mathbf{R}), H)$ the space of Hilbert-Schmidt operators from $L^2(\mathbf{R})$ into H. Its norm is given by

$$\|\Phi\|^2_{L_2^0(L^2(\mathbf{R}),H)} = \sum_{j\in\mathbf{N}} |\Phi e_j|_H^2.$$

When $H = H^s(\mathbf{R})$, $L_2^0(L^2(\mathbf{R}), H^s(\mathbf{R})) = L_2^{0,s}$.

The main results of this paper are as follows:

Theorem A Let $0 < \alpha \leq 1$, $u_0(x) \in L^2(\Omega; H^s(\mathbf{R}))$ be $\mathcal{F}_0$-measurable with $s \geq \frac{1}{2} - \frac{\alpha}{4}$ and $\Phi \in L_2^{0,\frac{\alpha+1}{2}}$. Then the Cauchy problem for (1.1) locally well-posed with $k = 2$.

Theorem B Let $0 < \alpha \leq 1$, $u_0 \in L^2(\Omega; H^{\frac{\alpha+1}{2}}(\mathbf{R})) \cap L^{\frac{2(2+3\alpha)}{\alpha}}(\Omega; L^2(\mathbf{R}))$ be $\mathcal{F}_0$-measurable and $\Phi \in L_2^{0,\frac{\alpha+1}{2}}$. Then the Cauchy problem for (1.1) possesses a unique global solution $u \in L^2(\Omega; H^{\frac{\alpha+1}{2}}(\mathbf{R}))$ with $k = 2$.

The rest of the paper is organized as follows. In section 2, some key interpolate inequalities and preliminary estimates are established. The section 3 is devoted to bilinear estimate by Fourier restriction norm method. We will show the trilinear estimate and local well-posedness of Cauchy problem in section 4. The global well-posedness of Cauchy problem is established in section 5.

2 Preliminaries

Lemma 2.1. *Let $\theta \in [0, 1]$ and $W_\gamma(t)u_0(x) = \int_R e^{i(t\phi(\xi)+x\xi)}|\phi''(\xi)|^{\frac{\gamma}{2}}\mathcal{F}_x u_0(\xi)d\xi$. Then*

$$\|W_{\frac{\theta}{2}}(t)u_0\|_{L_t^q L_x^p} \leq C\|u_0\|_{L_x^2},$$

where $(p, q) = (\frac{2}{1-\theta}, \frac{4}{\theta})$.

For the proof of Lemma 2.1, we refer the readers to Theorem 2.1 of [21].

Lemma 2.2. *Let $b = \frac{1}{2} + \epsilon$, then*

$$\|u\|_{L^4_{xt}} \leq C\|u\|_{X_{0,\frac{\alpha+3}{2(\alpha+2)}(\frac{1}{2}+\epsilon)}} \tag{2.1}$$

and

$$\left\|D_x^{\frac{\alpha}{8}} u\right\|_{L^6_{xt}} \leq C\|u\|_{X_{0,\frac{3}{4}b}}. \tag{2.2}$$

Proof. Let $\theta = \frac{2}{3}$, it follows from lemma 1 that

$$\left\| \int_R e^{it\phi(\xi)+ix\xi} |\phi''(\xi)|^{\frac{1}{6}} \mathcal{F}_x u_0(\xi) d\xi \right\|_{L^6_{xt}} \leq C\|u_0\|_{L^2_\xi}.$$

where $|\phi| = |\xi|^{\alpha+1}$, $|\phi''| = c|\xi|^\alpha$, then

$$\left\| \int_R e^{it\phi(\xi)+ix\xi} |\xi|^{\frac{\alpha}{6}} \mathcal{F}_x u_0(\xi) d\xi \right\|_{L^6_{xt}} \leq C\|u_0\|_{L^2_\xi}.$$

Due to $\|f\|_{L^{2\alpha+6}_{xt}} \leq C\|D^\gamma_x D^\gamma_t f\|_{L^6_{xt}}$ where $\gamma = \frac{\alpha}{6(\alpha+3)}$. Then

$$\begin{aligned} \|W(t)u_0(x)\|_{L^{2\alpha+4}_{xt}} &= C\left\| \int_R e^{i(t\phi+x\xi)} \mathcal{F}_x u_0(\xi) d\xi \right\|_{L^{2\alpha+4}_{xt}} \\ &\leq C\left\| D^\gamma_x D^\gamma_t \int_R e^{i(t\phi+x\xi)} \mathcal{F}_x u_0(\xi) d\xi \right\|_{L^6_{xt}} \\ &= C\left\| \int_R e^{i(t\phi+x\xi)} |\xi|^{\frac{\alpha}{6}} \mathcal{F}_x u_0(\xi) d\xi \right\|_{L^6_{xt}} \leq C\|u_0\|_{L^2_x}. \end{aligned} \tag{2.3}$$

By a standard argument, it follows from $\|W(t)u_0(x)\|_{L^{2\alpha+6}_{xt}} \leq C\|u_0\|_{L^2_x}$ that

$$\|u(x)\|_{L^{2\alpha+6}_{xt}} \leq C\|u\|_{X_{0,\frac{1}{2}+\epsilon}}. \tag{2.4}$$

By using the Plancherel identity, we have that

$$\|u\|_{L^2_{xt}} = C\|u\|_{X_{0,0}}. \tag{2.5}$$

Interpolating (2.4) with (2.5) yields

$$\|u\|_{L^4_{xt}} \leq C\|u\|_{X_{0,\frac{\alpha+3}{2(\alpha+2)}(\frac{1}{2}+\epsilon)}}. \tag{2.6}$$

From (2.3), by using a standard proof, we have that

$$\|D^{\frac{\alpha}{6}}_x u\|_{L^6_{xt}} \leq C\|u\|_{X_{0,b}}. \tag{2.7}$$

Interpolating (2.7) with (2.5) yields

$$\|D^{\frac{\alpha}{8}}_x u\|_{L^4_{xt}} \leq C\|u\|_{X_{0,\frac{3}{4}b}}. \tag{2.8}$$

Hence, the proof of Lemma 2.2 is completed. □

Lemma 2.3. *Let $b = \frac{1}{2}+\epsilon$. Then, for $0 \leq s \leq \frac{1}{2}$, we have that*

$$\|I^s(u_1,u_2)\|_{L^2_{xt}} \leq C\prod_{j=1}^{2} \|u_j\|_{X_{0,\frac{\alpha+3+2(\alpha+1)s}{2(\alpha+2)}b}}, \tag{2.9}$$

where

$$\mathscr{F} I^s(u_1,u_2)(\xi,\tau) = \int_{\substack{\xi = \xi_1+\xi_2 \\ \tau = \tau_1+\tau_2}} ||\xi_1|^{\alpha+1} - |\xi_2|^{\alpha+1}|^s \mathscr{F} u_1(\xi_1,\tau_1)\mathscr{F} u_2(\xi_2,\tau_2)\, d\xi_1 d\tau_1.$$

Proof. Let $F_j(\xi_j,\tau_j)=\langle\sigma_j\rangle^{\frac{\alpha+3+2(\alpha+1)s}{2\alpha+4}b}\mathscr{F}u_j(\xi_j,\tau_j)(j=1,2)$. To prove Lemma 2.3, by the Plancherel identity, it suffices to prove that

$$\left\|\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}}||\xi_1|^{\alpha+1}-|\xi_2|^{\alpha+1}|^s\frac{F_1}{\langle\sigma_1\rangle^{\frac{\alpha+3+2(\alpha+1)s}{2\alpha+2}b}}\frac{F_2}{\langle\sigma_2\rangle^{\frac{\alpha+3+2(\alpha+1)s}{2\alpha+2}b}}d\xi_1d\tau_1\right\|_{L^2_{\xi\tau}}\leq C\prod_{j=1}^{2}\|F_j\|_{L^2_{\xi\tau}}. \quad (2.10)$$

Assume that $b_1=\frac{\alpha+3+2(\alpha+1)s}{2\alpha+4}b$. By using the Young inequality, since $0<s<\frac{1}{2}$, we have that

$$\begin{aligned}&||\xi_1|^{\alpha+1}-|\xi_2|^{\alpha+1}|^s\langle\sigma_1\rangle^{-b_1}\langle\sigma_2\rangle^{-b_1}\\&=||\xi_1|^{\alpha+1}-|\xi_2|^{\alpha+1}|^s\langle\sigma_1\rangle^{-2bs}\langle\sigma_2\rangle^{-2bs}\langle\sigma_1\rangle^{-(b_1-2bs)}\langle\sigma_2\rangle^{-(b_1-2bs)}\\&\leq 2s||\xi_1|^{\alpha+1}-|\xi_2|^{\alpha+1}|^{1/2}\langle\sigma_1\rangle^{-b}\langle\sigma_2\rangle^{-b}+(1-2s)\langle\sigma_1\rangle^{-\frac{\alpha+3}{2\alpha+4}b}\langle\sigma_2\rangle^{-\frac{\alpha+3}{2\alpha+4}b}\\&\leq ||\xi_1|^{\alpha+1}-|\xi_2|^{\alpha+1}|^{1/2}\langle\sigma_1\rangle^{-b}\langle\sigma_2\rangle^{-b}+\langle\sigma_1\rangle^{-\frac{\alpha+3}{2\alpha+4}b}\langle\sigma_2\rangle^{-\frac{\alpha+3}{2\alpha+4}b}.\end{aligned} \quad (2.11)$$

By using (2.11), Plancherel identity, Lemma 3.1 in [18], we have that

$$\begin{aligned}&\left\|\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}}||\xi_1|^{\alpha+1}-|\xi_2|^{\alpha+1}|^s\frac{F_1}{\langle\sigma_1\rangle^{\frac{\alpha+3+2(\alpha+1)s}{2\alpha+4}b}}\frac{F_2}{\langle\sigma_2\rangle^{\frac{\alpha+3+2(\alpha+1)s}{2\alpha+2}b}}d\xi_1d\tau_1\right\|_{L^2_{\xi\tau}}\\&\leq\left\|\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}}||\xi_1|^{\alpha+1}-|\xi_2|^{\alpha+1}|^{1/2}\prod_{j=1}^{2}\frac{F_j}{\langle\sigma_j\rangle^b}d\xi_1d\tau_1\right\|_{L^2_{\xi\tau}}+\left\|\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}}\prod_{j=1}^{2}\frac{F_1}{\langle\sigma_j\rangle^{\frac{\alpha+3}{2\alpha+4}b}}d\xi_1d\tau_1\right\|_{L^2_{\xi\tau}}\\&\leq C\prod_{j=1}^{2}\left\|\mathscr{F}^{-1}\left(\frac{F_j}{\langle\sigma_j\rangle^b}\right)\right\|_{X_{0,b}}+C\prod_{j=1}^{2}\left\|\mathscr{F}^{-1}\left(\frac{F_j}{\langle\sigma_j\rangle^{\frac{\alpha+3}{2\alpha+4}b}}\right)\right\|_{X_{0,\frac{\alpha+3}{2\alpha+4}b}}\\&\leq C\prod_{j=1}^{2}\|F_j\|_{L^2_{\xi\tau}}.\end{aligned} \quad (2.12)$$

The proof of Lemma 2.3 is completed. □

Lemma 2.4. *Let $u_0\in H^s(\mathbf{R})$, $c>1/2, 0<b<1/2$. Then for $t\in[0,T]$, $W(t)u_0\in X^T_{s,\,c}$ and there is a constant $k_2>0$ such that*

$$\|U(t)u_0\|_{X^T_{s,c}}\leq k_2\|u_0\|_{H^s}. \quad (2.13)$$

There is a constant $c>0$ such that for $t\in[0,1]$ and $f\in X^T_{s,\,b}$,

$$\left\|\int_0^T U(t-s)f(s)ds\right\|_{X^T_{s,b}}\leq CT^{1-2b}\|f\|_{X_{-b,s}}. \quad (2.14)$$

For the proof of Lemma 2.4, we refer the readers to Lemma 3.1 of [3].

Lemma 2.5. *Let*

$$\overline{u} = \int_0^t U(t-s)\Phi dW(s)$$

and $\Phi \in L_2^{0,s}$*, for* $t \in [0,T]$*, we have*

$$E(\sup_{t\in[0,T]} \|\overline{u}\|_{H^s}^2) \le 38T\|\Phi\|_{L_2^{0,s}}^2. \tag{2.15}$$

Lemma 2.5 can be proved similarly to Proposition 3.1 of [2].

Lemma 2.6. *Let*

$$\overline{u} = \int_0^t U(t-s)\Phi dW(s)$$

and $\Phi \in L_2^{0,s}$*, for* $t \in [0,T]$*, we have* $\Psi\overline{u} \in L^2(\Omega; X_{s,b})$ *and*

$$E(\sup_{t\in[0,T]} \|\Psi\overline{u}\|_{X_{s,b}}^2) \le M(b,\Psi)\|\Phi\|_{L_2^{0,s}}^2. \tag{2.16}$$

Lemma 2.6 can be proved similarly to Proposition 2.1 of [3].

3 Bilinear estimate

Theorem 3.1. *For all* u, v *on* $\mathbf{R}\times\mathbf{R}$ *and* $0 < \alpha \le 1$*,* $0 < \epsilon \le \frac{\alpha}{2(3\alpha+8)}$ *and* $b = \frac{1}{2} - \epsilon$*, we have*

$$\|u_1 u_2\|_{L^2} \le C\|u_1\|_{X_{-\frac{1}{2},b}}\|u_2\|_{X_{\frac{1}{2}-\frac{\alpha}{4},b}}. \tag{3.1}$$

Proof. Define

$$F_1(\xi_1,\tau_1) = \langle\xi_1\rangle^{-1/2}\langle\sigma_1\rangle^b \mathcal{F}u_1(\xi_1,\tau_1), \quad F_2(\xi_2,\tau_2) = \langle\xi_2\rangle^{\frac{1}{2}-\frac{\alpha}{4}}\langle\sigma_2\rangle^b \mathcal{F}u(\xi_2,\tau_2),$$
$$\sigma_j = \tau_j - |\xi_j|^{\alpha+1}\xi_j,\ j = 1,2.$$

To obtain (4.4), it suffices to prove that

$$\int_{\mathbf{R}^2}\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} K_1(\xi_1,\tau_1,\xi,\tau)|F|\prod_{j=1}^{2}|F_j|d\xi_1 d\tau_1 d\xi d\tau \le C\|F\|_{L^2_{\xi\tau}}\prod_{j=1}^{2}\|F_j\|_{L^2_{\xi\tau}}, \tag{3.2}$$

where

$$K_1(\xi_1,\tau_1,\xi,\tau) = \frac{\langle\xi_1\rangle^{1/2}\langle\xi_2\rangle^{\frac{\alpha}{4}-\frac{1}{2}}}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}.$$

Without loss of generality, we assume that $F \geq 0, F_j \geq 0 (j = 1, 2)$. Obviously,

$$\{(\xi_1, \tau_1, \xi, \tau) \in \mathrm{R}^4, \xi = \sum_{j=1}^{2} \xi_j, \tau = \sum_{j=1}^{2} \tau_j\} \subset \sum_{j=1}^{6} \Omega_j$$

where

$$\Omega_1 = \{(\xi_1, \tau_1, \xi, \tau) \in \mathrm{R}^4, \xi = \sum_{j=1}^{2} \xi_j, \tau = \sum_{j=1}^{2} \tau_j, |\xi_1| \leq |\xi_2| \leq 6\},$$

$$\Omega_2 = \{(\xi_1, \tau_1, \xi, \tau) \in \mathrm{R}^4, \xi = \sum_{j=1}^{2} \xi_j, \tau = \sum_{j=1}^{2} \tau_j, |\xi_2| \geq 6, |\xi_2| \gg |\xi_1|\},$$

$$\Omega_3 = \{(\xi_1, \tau_1, \xi, \tau) \in \mathrm{R}^4, \xi = \sum_{j=1}^{2} \xi_j, \tau = \sum_{j=1}^{2} \tau_j, |\xi_2| \geq 6, |\xi_2| \sim |\xi_1|\},$$

$$\Omega_4 = \{(\xi_1, \tau_1, \xi, \tau) \in \mathrm{R}^4, \xi = \sum_{j=1}^{2} \xi_j, \tau = \sum_{j=1}^{2} \tau_j, |\xi_2| \leq |\xi_1| \leq 6\},$$

$$\Omega_5 = \{(\xi_1, \tau_1, \xi, \tau) \in \mathrm{R}^4, \xi = \sum_{j=1}^{2} \xi_j, \tau = \sum_{j=1}^{2} \tau_j, |\xi_1| \geq 6, |\xi_1| \gg |\xi_2|\},$$

$$\Omega_6 = \{(\xi_1, \tau_1, \xi, \tau) \in \mathrm{R}^4, \xi = \sum_{j=1}^{2} \xi_j, \tau = \sum_{j=1}^{2} \tau_j, |\xi_1| \geq 6, |\xi_1| \geq |\xi_2|, |\xi_1| \sim |\xi_2|\},$$

We define

$$f_j = \mathcal{F}^{-1} \frac{F_j}{\langle \sigma_j \rangle^b}, j = 1, 2.$$

The integrals corresponding to $\Omega_j (1 \leq j \leq 6, j \in \mathrm{N}^+)$ will be denoted by $J_k (1 \leq k \leq 6, k \in \mathrm{N}^+)$ in (3.2), respectively.
(1). $\Omega_1 = \{(\xi_1, \tau_1, \xi, \tau) \in \mathrm{R}^4, \xi = \sum_{j=1}^{2} \xi_j, \tau = \sum_{j=1}^{2} \tau_j, |\xi_1| \leq |\xi_2| \leq 6\}$. In this subregion, we have that

$$K_1(\xi_1, \tau_1, \xi, \tau) \leq \frac{C}{\prod_{j=1}^{2} \langle \sigma \rangle^b}.$$

By using the Plancherel identity and the Hölder inequality and $\frac{\alpha+3}{2(\alpha+2)}(\frac{1}{2}+\epsilon) < \frac{1}{2}-\epsilon$, we have that

$$
\begin{aligned}
J_1 &\leq C\int_{\mathbf{R}^2}\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} \frac{F\prod_{j=1}^2 F_j}{\prod_{j=1}^2\langle\sigma_j\rangle^b} d\xi_1 d\tau_1 d\xi d\tau \\
&\leq C\int_{\mathbf{R}^2}\mathcal{F}^{-1}(F) f_1 f_2 dxdt \leq C\|\mathcal{F}^{-1}(F)\|_{L^2_{xt}}\prod_{j=1}^2\|f_j\|_{L^4_{xt}} \\
&\leq C\|F\|_{L^2_{\xi\tau}}\prod_{j=1}^2\|f_j\|_{X_{0,\frac{\alpha+3}{2(\alpha+2)}(\frac{1}{2}+\epsilon)}} \\
&\leq C\|F\|_{L^2_{\xi\tau}}\prod_{j=1}^2\|F_j\|_{L^2_{\xi\tau}}. \qquad (3.3)
\end{aligned}
$$

(2).$\Omega_2 = \{(\xi_1,\tau_1,\xi,\tau)\in \mathrm{R}^4, \xi=\sum_{j=1}^2\xi_j, \tau=\sum_{j=1}^2\tau_j, |\xi_2|\geq 6, |\xi_2|\gg|\xi_1|\}$.
If $|\xi_1|\leq 1$, we have that

$$
K_1(\xi_1,\tau_1,\xi,\tau) \leq \frac{C}{\prod_{j=1}^2\langle\sigma_j\rangle^b}
$$

This case can be proved similarly to Ω_1.
If $|\xi_1|\geq 1$, we have that

$$
K_1(\xi_1,\tau_1,\xi,\tau) \leq C\frac{|\xi_2|^{\frac{\alpha}{4}}}{\prod_{j=1}^2\langle\sigma_j\rangle^b} \leq C\frac{||\xi_2|^{\alpha+1}-|\xi_1|^{\alpha+1}|^{\frac{\alpha}{4(\alpha+1)}}}{\prod_{j=1}^2\langle\sigma_j\rangle^b}.
$$

By using the Cauchy-Schwartz inequality and Lemma 2.3 as well as $0<\epsilon<\frac{1}{14}$, we have that

$$
\begin{aligned}
J_2 &\leq C\int_{\mathbf{R}^2}\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} \frac{||\xi_2|^{\alpha+1}-|\xi_1|^{\alpha+1}|^{\frac{\alpha}{4(\alpha+1)}} F\prod_{j=1}^2 F_j}{\prod_{j=1}^2\langle\sigma_j\rangle^b} d\xi_1 d\tau_1 d\xi d\tau \\
&\leq C\|F\|_{L^2_{\xi\tau}}\left\|\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} \frac{||\xi_2|^{\alpha+1}-|\xi_1|^{\alpha+1}|^{\frac{\alpha}{4(\alpha+1)}} F\prod_{j=1}^2 F_j}{\prod_{j=1}^2\langle\sigma_j\rangle^b} d\xi_1 d\tau_1\right\|_{L^2_{\xi\tau}} \\
&\leq C\|F\|_{L^2_{\xi\tau}}\prod_{j=1}^2\|F_j\|_{L^2_{\xi\tau}}.
\end{aligned}
$$

(3). $\Omega_3 = \{(\xi_1,\tau_1,\xi,\tau)\in \mathrm{R}^4, \xi=\sum_{j=1}^2\xi_j, \tau=\sum_{j=1}^2\tau_j, |\xi_2|\geq 6, |\xi_2|\sim|\xi_1|\}$. In this subregion, we have that

$$
K_1(\xi_1,\tau_1,\xi,\tau) \leq C\frac{|\xi_2|^{\frac{\alpha}{4}}}{\prod_{j=1}^2\langle\sigma_j\rangle^b} \leq C\frac{\prod_{j=1}^2|\xi_j|^{\frac{\alpha}{8}}}{\prod_{j=1}^2\langle\sigma_j\rangle^b}
$$

By using the Plancherel identity and the Cauchy-Schwartz inequality as well as $\frac{3}{4}(\frac{1}{2}+\epsilon) < \frac{1}{2}-\epsilon$, we have that

$$\begin{aligned}
J_3 &\leq C\int_{\mathbf{R}^2}\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} \frac{\prod_{j=1}^{2}|\xi_j|^{\frac{\alpha}{8}}F\prod_{j=1}^{2}F_j}{\prod_{j=1}^{2}\langle\sigma_j\rangle^b}d\xi_1 d\tau_1 d\xi d\tau \\
&\leq C\|F\|_{L^2_{\xi\tau}}\prod_{j=1}^{2}\left\|D_x^{\frac{\alpha}{8}}\mathcal{F}^{-1}\left(\frac{F_j}{\langle\sigma_j\rangle^b}\right)\right\|_{L^4_{xt}} \\
&\leq C\|F\|_{L^2_{\xi\tau}}\prod_{j=1}^{2}\|F_j\|_{L^2_{\xi\tau}}. \qquad (3.4)
\end{aligned}$$

(4). $\Omega_4 = \{(\xi_1,\tau_1,\xi,\tau)\in \mathrm{R}^4, \xi=\sum_{j=1}^{2}\xi_j, \tau=\sum_{j=1}^{2}\tau_j, |\xi_2|\leq|\xi_1|\leq 6\}$. In this subregion, we have that

$$K_1(\xi_1,\tau_1,\xi,\tau)\leq \frac{C}{\prod_{j=1}^{2}\langle\sigma_j\rangle^b}.$$

Thus subregion can be proved similarly to Ω_1.
(5). $\Omega_5 = \{(\xi_1,\tau_1,\xi,\tau)\in \mathrm{R}^4, \xi=\sum_{j=1}^{2}\xi_j, \tau=\sum_{j=1}^{2}\tau_j, |\xi_1|\geq 6, |\xi_1|\gg|\xi_2|\}$. In this subregion, we have that

$$K_1(\xi_1,\tau_1,\xi,\tau)\leq C\frac{||\xi_1|^{\alpha+1}-|\xi_2|^{\alpha+1}|^{\frac{1}{2(\alpha+1)}}}{\prod_{j=1}^{2}\langle\sigma_j\rangle^b}.$$

By using the Cauchy-Schwartz inequality and using Lemma 2.3 as well as $\frac{\alpha+4}{2(\alpha+2)}(\frac{1}{2}+\epsilon)\leq\frac{1}{2}-\epsilon$, we have that

$$\begin{aligned}
J_5 &\leq C\int_{\mathbf{R}^2}\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} \frac{||\xi_2|^{\alpha+1}-|\xi_1|^{\alpha+1}|^{\frac{1}{2(\alpha+1)}}F\prod_{j=1}^{2}F_j}{\prod_{j=1}^{2}\langle\sigma_j\rangle^b}d\xi_1 d\tau_1 d\xi d\tau \\
&\leq C\|F\|_{L^2_{\xi\tau}}\left\|\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} \frac{||\xi_2|^{\alpha+1}-|\xi_1|^{\alpha+1}|^{\frac{1}{2(\alpha+1)}}F\prod_{j=1}^{2}F_j}{\prod_{j=1}^{2}\langle\sigma_j\rangle^b}d\xi_1 d\tau_1\right\|_{L^2_{\xi\tau}} \\
&\leq C\|F\|_{L^2_{\xi\tau}}\prod_{j=1}^{2}\|F_j\|_{L^2_{\xi\tau}}.
\end{aligned}$$

(6). $\Omega_6 = \{(\xi_1,\tau_1,\xi,\tau)\in \mathrm{R}^4, \xi=\sum_{j=1}^{2}\xi_j, \tau=\sum_{j=1}^{2}\tau_j, |\xi_1|\geq 6, |\xi_1|\geq|\xi_2|, |\xi_1|\sim|\xi_2|\}$. In this subregion, $|\xi_1|\sim|\xi_2|$.

This subregion can be proved similarly to Ω_3.

Thus, We have completed the proof of Theorem 3.1. $\square$

4 Local well-posedness

In this section, we will establish two new trilinear estimates which play a crucial role in establishing the local well-posedness of solution, and then we will show the local well-posedness of the Cauchy problem (1.1) by Banach Fixed Point theorem.

We will establish the Lemma 4.1 with the aid of the idea in [38]. Let $Z = \mathbf{R}$ and $\Gamma_k(Z)$ denote the hyperplane in $\mathbf{R}^k$

$$\Gamma_k(Z) := \left\{(\xi_1, \cdots, \xi_k) \in Z^k, \xi_1 + \cdots + \xi_k = 0\right\}$$

endowed with the induced measure

$$\int_{\Gamma_k(Z)} f := \int_{Z^{k-1}} f(\xi_1, \cdots, \xi_{k-1}, -\xi_1 - \cdots - \xi_{k-1}) d\xi_1 \cdots d\xi_k.$$

A function $m : \Gamma_k(Z) \to C$ is said to be a $[k; Z]$-multiplier, and we define the norm $\|m\|_{[k;Z]}$ to be the best constant such that the inequality

$$\left|\int_{\Gamma_k(Z)} m(\xi) \prod_{j=1}^{k} f_j(\xi_j)\right| \leq \|m\|_{[k;Z]} \prod_{j=1}^{k} \|f_j\|_{L^2}.$$

holds for all test function f_j on Z.

Lemma 4.1. *Let $s_0 = \frac{1}{2} - \frac{\alpha}{4}$, $b = \frac{1}{2} - \epsilon$. Then*

$$\|\partial_x(u_1u_2u_3)\|_{X_{s_0,-b}} \leq C \prod_{j=1}^{3} \|u_j\|_{X_{s_0,b}}. \tag{4.1}$$

Proof. By duality, Plancherel identity and the definition, to obtain (4.4), it suffices to prove that

$$\left\|\frac{(\sum_{j=1}^{3} \xi_j)\langle\xi_4\rangle^{\frac{1}{2}-\frac{\alpha}{4}}}{\prod_{j=1}^{4}\langle\tau_j - w(\xi_j)\rangle^{\frac{1}{2}-\epsilon} \prod_{j=1}^{3}\langle\xi_j\rangle^{\frac{1}{2}-\frac{\alpha}{4}}}\right\|_{[4;\mathbf{R}\times\mathbf{R}]} \leq C. \tag{4.2}$$

By using the symmetry and

$$\langle\xi_4\rangle^{\frac{3}{2}-\frac{\alpha}{4}} \leq C\langle\xi_4\rangle^{\frac{1}{2}} \left[\sum_{j=1}^{3}\langle\xi_j\rangle^{1-\frac{\alpha}{4}}\right]$$

resulting from

$$|\xi_1 + \xi_2 + \xi_3| \leq \langle\xi_4\rangle,$$

to obtain (4.2), it suffices to prove

$$\left\|\frac{\langle\xi_4\rangle^{1/2}\langle\xi_2\rangle^{1/2}}{\langle\xi_1\rangle^{\frac{1}{2}-\frac{\alpha}{4}}\langle\xi_3\rangle^{\frac{1}{2}-\frac{\alpha}{4}} \prod_{j=1}^{4}\langle\tau_j - w(\xi_j)\rangle^{\frac{1}{2}-\epsilon}}\right\|_{[4;\mathbf{R}\times\mathbf{R}]} \leq C. \tag{4.3}$$

(4.3) follows from $TT^\star$ identity in Lemma 3.7 of [38] and Theorem 3.1.

The proof of lemma 4.1 has been completed. $\square$

Lemma 4.2. *Let $s \geq s_0 = \frac{1}{2} - \frac{\alpha}{4}$, $b = \frac{1}{2} - \epsilon$. Then*

$$\|\partial_x(u_1u_2u_3)\|_{X_{s,-b}} \leq C \prod_{j=1}^{3} \|u_j\|_{X_{s,b}}. \tag{4.4}$$

Proof. (4.4) is equivalent to the following inequality

$$\int_{\mathbf{R}^2} \int_{\substack{\xi = \xi_1 + \xi_2 + \xi_3 \\ \tau = \tau_1 + \tau_2 + \tau_3}} \frac{|\xi| \langle \xi \rangle^s F \prod_{j=1}^{3} F_j}{\langle \sigma \rangle^b \prod_{j=1}^{3} \langle \xi_j \rangle^s \langle \sigma_j \rangle^b} d\xi_1 d\tau_1 d\xi_2 d\tau_2 d\xi d\tau \leq C \|F\|_{L^2_{\xi\tau}} \prod_{j=1}^{3} \|F_j\|_{L^2_{\xi\tau}}. \tag{4.5}$$

Since

$$\langle \xi \rangle^{s-s_0} \leq C \prod_{j=1}^{3} \langle \xi_j \rangle^{s-s_0}, \tag{4.6}$$

(4.5) is equivalent to the following inequality

$$\int_{\mathbf{R}^2} \int_{\substack{\xi = \xi_1 + \xi_2 + \xi_3 \\ \tau = \tau_1 + \tau_2 + \tau_3}} \frac{|\xi| \langle \xi \rangle^{s_0} F \prod_{j=1}^{3} F_j}{\langle \sigma \rangle^b \prod_{j=1}^{3} \langle \xi_j \rangle^{s_0} \langle \sigma_j \rangle^b} d\xi_1 d\tau_1 d\xi_2 d\tau_2 d\xi d\tau \leq C \|F\|_{L^2_{\xi\tau}} \prod_{j=1}^{3} \|F_j\|_{L^2_{\xi\tau}}, \tag{4.7}$$

which is just the lemma 4.1. Hence, the proof of lemma 4.2 is completed. □

Following the method of [3], combining Lemma 4.2, Lemmas 2.4, 2.5 with the Banach Fixed Point Theorem, we have the following local well-posedness of Cauchy problem (1.1).

Theorem 4.1. *Let $0 < \alpha \leq 1$, $u_0(x) \in L^2(\Omega; H^s(\mathbf{R}))$ be $\mathcal{F}_0$-measurable with $s \geq \frac{1}{2} - \frac{\alpha}{4}$ and $\Phi \in L_2^{0, \frac{\alpha+1}{2}}$. Then the Cauchy problem for (1.1) locally well-posed with $k = 2$.*

5 Global Well-posedness

In this section, we always assume that $0 < \alpha \leq 1$, $u_0 \in L^2(\Omega; H^{\frac{\alpha+1}{2}}(\mathbf{R})) \cap L^{\frac{2(2+3\alpha)}{\alpha}}(\Omega; L^2(\mathbf{R}))$ be $\mathcal{F}_0$-measurable and $\Phi \in L_2^{0, \frac{\alpha+1}{2}}$. In order to obtain the global well-posedness, we follow the argument given by Bouard and Debussche in [3], in which, they established the global well-posedness for stochastic Korteweg-de Vries equation driven by white noise, we also refer [36] to the global well-posedness for stochastic KVD-Burgers equation.

Notice that the deterministic equation (1.3) possesses two important conservation laws:

$$\frac{1}{2} \int_{\mathbf{R}} u^2 dx, \qquad \frac{1}{2} \int_{\mathbf{R}} (D_x^{\frac{\alpha+1}{2}} u)^2 dx - \frac{1}{k+1} \int_{\mathbf{R}} u^{k+1} dx.$$

Let $(\Phi_m)_{m \in \mathbf{N}}$ be a sequence in $L_2^{0,4}$ such that $\Phi_m \longrightarrow \Phi$ in $L_2^{0, \frac{\alpha+1}{2}}$ and let $(u_{0m})_{m \in \mathbf{N}}$ be a sequence in $H^3(\mathbf{R})$ such that $u_{0m} \longrightarrow u_0$ in $L^2(\Omega; H^{\frac{\alpha+1}{2}}(\mathbf{R})) \cap L^{\frac{2(2+3\alpha)}{\alpha}}(\Omega; L^2(\mathbf{R}))$ and in $H^{\frac{\alpha+1}{2}}(\mathbf{R})$ a.s..

Lemma 5.1. *For sufficiently large m, then there exists a unique solution u_m* P-*a.s in*

$$L^\infty(0,T;H^{\frac{\alpha+1}{2}}(\mathbf{R}))$$

of

$$du_m + (D_x^{\alpha+1}u_{mx} - u_m^2 u_{mx}) = \Phi_m dW, \tag{5.1}$$
$$u_m(0) = u_{0m} \tag{5.2}$$

For any $T > 0$.

Proof. Define $v_m = u_m - \overline{u}_m$, then v_m satisfies

$$v_{mt} + D_x^{\alpha+1}v_{mx} - (v_m + \overline{u}_m)^2\partial_x(v_m + \overline{u}_m) = 0, \tag{5.3}$$
$$v_m(0) = u_{0m}. \tag{5.4}$$

By using Lemmas 2.4-2.6 and Lemma 4.2 as well as the fixed point point, we know that (5.3)-(5.4) have a local solution $u_m \in L^\infty(0,T_0(\omega),H^{\frac{\alpha+1}{2}}(\mathbf{R}))$ a.s. for $a.s.u_{0m} \in H^{\frac{\alpha+1}{2}}$, where $T_0(\omega)$ is the lifespan of the local solution. It follows from Lemma 2.5 that

$$\overline{u}_m = \int_0^t U(t-s)\Phi_m dW(s) \longrightarrow \int_0^t U(t-s)\Phi dW(s) \tag{5.5}$$

in $L^2(\Omega;H^4(\mathbf{R}))$ and $\overline{u}_m$ is in $L^\infty(0,T;H^4)$ a.s. From (5.5), we have that there exists a subsequence, still denoted by $\overline{u}_m$, such that

$$\overline{u}_m \longrightarrow \overline{u} \tag{5.6}$$

in $H^4(\mathbf{R})$ a.s.

We claim that the sequence u_m is bounded in $L^2(\Omega;L^\infty(0,T;H^{\frac{\alpha+1}{2}}(\mathbf{R}))$ when $u_0 \in L^2(\Omega;H^{\frac{\alpha+1}{2}}(\mathbf{R}))\cap L^{\frac{2(2+3\alpha)}{\alpha}}(\Omega;L^2(\mathbf{R}))$ for any $T > 0$.

In fact, denote

$$I(u_m) = \frac{1}{2}\int_{\mathbf{R}}(D_x^{\frac{\alpha+1}{2}}u_m)^2dx - \frac{1}{4}\int_{\mathbf{R}}u_m^4 dx. \tag{5.7}$$

Applying the Itô formula to $I(u_m)$ yields

$$I(u_m) = I(u_{m0}) - \int_0^t (D_x^{\alpha+1}u_m - u_m^3, \Phi dW(s)) + \frac{1}{2}\int_0^t Tr\,(I''(u_m)\Phi_m\Phi_m^\star)\,ds. \tag{5.8}$$

with

$$I''(u_m)\phi = D_x^{\alpha+1}\phi - 3u_m^2\phi.$$

Let $(e_i)_{i\in N}$ be an orthonormal basis of $L^2(R)$, by using $H^{\frac{\alpha+1}{2}} \hookrightarrow L^\infty$, we have that

$$Tr\left(I''(u_m)\Phi_m\Phi_m^\star\right) = -\sum_{j\in \mathrm{N}} \int_{\mathbf{R}} \left[D_x^{\alpha+1}(\Phi_m e_j)\Phi_m e_j + 3u_m^2(\Phi_m e_j)^2\right] dx$$
$$\leq \sum_{j\in \mathrm{N}} \left(\left\|D_x^{\frac{\alpha+1}{2}}(\Phi_m e_j)\right\|_{L^2}^2 + 3\|u_m\|_{L^2}^2 \|\Phi_m e_j\|_{L^\infty}^2\right) \leq 3\|\Phi_m\|^2_{L_2^{0,\frac{\alpha+1}{2}}} \left[\|u_m\|_{L^2}^2 + 1\right]. \quad (5.9)$$

We derive from (5.8) that

$$\mathrm{E}\left(\sup_{t\in[0,T]} \frac{1}{2}\int_0^t Tr\left(I''(u_m)\Phi_m\Phi_m^\star\right) ds\right) \leq C\mathrm{E}\left(\sup_{t\in[0,T]} \|u_m\|_{L^2}^4\right) + CT^2\|\Phi_m\|^4_{L_2^{0,\frac{\alpha+1}{2}}} + C. \quad (5.10)$$

Applying the martingale inequality (Theorem 3.14 of [12]) yields

$$\mathrm{E}\left(\sup_{t\in[0,T]} -\int_0^t (D_x^{\frac{\alpha+1}{2}} u_m - u_m^3, \Phi dW(s))\right) \leq 3\mathrm{E}\left(\left(\int_0^T \left|\Phi_m^\star\left(D_x^{\frac{\alpha+1}{2}} u_m - u_m^3\right)\right|^2 ds\right)^{1/2}\right). \quad (5.11)$$

By using the Sobolev embedding $H^{\frac{\alpha+1}{2}} \hookrightarrow L^\infty$ and interpolation Theorem, we obtain

$$\begin{aligned}\left|\Phi_m^\star\left(D_x^{\alpha+1}u_m - u_m^3\right)\right|^2 &= \sum_{j\in N}\left[(D_x^{\alpha+1}u_m, \Phi_m e_j) + (u_m^3, \Phi_m e_j)\right]^2 \\ &\leq C\sum_{j\in N}\left[\|u_m\|^2_{H^{\frac{\alpha+1}{2}}}\|\Phi_m e_j\|^2_{H^{\frac{\alpha+1}{2}}} + \|u_m\|_{L^3}^6\|\Phi_m e_j\|_{L^\infty}^2\right] \\ &\leq C\left[\|u_m\|^2_{H^{\frac{\alpha+1}{2}}} + \|u_m\|_{L^2}^{\frac{2(2+3\alpha)}{\alpha+1}}\|D_x^{\frac{\alpha+1}{2}}u_m\|_{L^2}^{\frac{2}{\alpha+1}}\right]\|\Phi_m\|^2_{L_2^{0,\frac{\alpha+1}{2}}}. \quad (5.12)\end{aligned}$$

Substituting (5.12) into (5.11), using Cauchy-Schwartz inequality and Young inequality, we deduce

$$\begin{aligned}&\mathrm{E}\left(\sup_{t\in[0,T]} -\int_0^t (D_x^{\alpha+1}u_m - u_m^3, \Phi dW(s))\right) \\ &\leq \frac{1}{8}\mathrm{E}\left(\sup_{t\in[0,T]} \|D_x^{\frac{\alpha+1}{2}}u_m\|_{L^2}^2\right) + C\mathrm{E}\left(\sup_{t\in[0,T]}\|u_m\|_{L^2}^{\frac{2(2+3\alpha)}{\alpha}}\right) + CT\|\Phi_m\|^2_{L_2^{0,\frac{\alpha+1}{2}}}. \quad (5.13)\end{aligned}$$

Applying Itô formula to $F(u_m) = \|u_m\|_{L^2}^{\frac{2(2+3\alpha)}{\alpha}}$ yields

$$\begin{aligned}&\|u_m\|_{L^2}^{\frac{2(2+3\alpha)}{\alpha}} \\ &= \|u_{0m}\|_{L^2}^{\frac{2(2+3\alpha)}{\alpha}} + \frac{2(2+3\alpha)}{\alpha}\int_0^t \|u_m\|_{L^2}^{\frac{2(2+2\alpha)}{\alpha}}(u_m, \Phi_m dW) + \frac{1}{2}\int_0^t Tr(F''(u_m)\Phi_m\Phi_m^\star)ds, \quad (5.14)\end{aligned}$$

where

$$F''(u_m)\phi = \frac{4(2+3\alpha)(2+2\alpha)}{\alpha^2}\|u_m\|_{L^2}^{\frac{2(2+\alpha)}{\alpha}}(u_m,\phi)u_m + \frac{2(2+3\alpha)}{\alpha}\|u_m\|_{L^2}^{\frac{2(2+\alpha)}{\alpha}}\phi.$$

By using a martingale inequality (Theorem 3.14 in [12]) and Young inequality, we have that

$$\begin{aligned}
&\mathrm{E}\left(\sup_{t\in[0,T]}\frac{2(2+3\alpha)}{\alpha}\int_0^t\|u_m\|_{L^2}^{\frac{2(2+2\alpha)}{\alpha}}(u_m,\Phi_m dW)\right)\\
&\le 3\mathrm{E}\left(\left(\int_0^T\|u_m\|_{L^2}^{\frac{4(2+2\alpha)}{\alpha}}\|\Phi_m^\star u_m\|_{L^2}^2\right)^{1/2}\right)\\
&\le \frac{1}{16}\mathrm{E}\left(\sup_{t\in[0,T]}\|u_m\|_{L^2}^{\frac{2(2+3\alpha)}{\alpha}}\right)+CT\|\Phi_m\|_{L_2^{0,0}}^{4\alpha+6}. \qquad (5.15)
\end{aligned}$$

Direct computation implies

$$\begin{aligned}
&Tr(F''(u_m)\Phi_m\Phi_m^\star)\\
&=\sum_{j\in N}\frac{4(2+3\alpha)(2+2\alpha)}{\alpha^2}\|u_m\|_{L^2}^{\frac{2(2+\alpha)}{\alpha}}(u_m,\Phi_m e_j)^2+\sum_{j\in N}\frac{2(2+3\alpha)}{\alpha}\|u\|_{L^2}^{\frac{2(2+2\alpha)}{\alpha}}\|\Phi_m e_j\|_{L^2}^2\\
&\quad\le\frac{2(2+3\alpha)(4+5\alpha)}{\alpha^2}\|u_m\|_{L^2}^{\frac{2(2+\alpha)}{\alpha}}\|\Phi_m\|_{L_2^{0,0}}^2\\
&\quad\le\frac{1}{2}\|u_m\|_{L^2}^{\frac{2(2+3\alpha)}{\alpha}}+C\|\Phi_m\|_{L_2^{0,0}}^{\frac{2+3\alpha}{\alpha}}. \qquad (5.16)
\end{aligned}$$

Combining (5.15), (5.16) with (5.14), we have that

$$\mathrm{E}\left(\sup_{t\in[0,T_0]}\|u_m(t)\|_{L^2}^{\frac{2(2+3\alpha)}{\alpha}}\right)\le C\mathrm{E}\left(\|u_{0m}\|_{L^2}^{\frac{2(2+3\alpha)}{\alpha}}\right)+C\|\Phi_m\|_{L_2^{0,0}}^{\frac{2+3\alpha}{\alpha}}+CT\|\Phi_m\|_{L_2^{0,0}}^{4\alpha+6}. \qquad (5.17)$$

By using interpolation Theorem, we have that

$$\|u_m\|_{L^4}^4\le C\|u_m\|_{L^2}^{\frac{2(2\alpha+1)}{\alpha+1}}\|D_x^{\frac{\alpha+1}{2}}u_m\|_{L^2}^{\frac{2}{\alpha+1}}\le C\|u_m\|_{L^2}^{\frac{2(2\alpha+1)}{\alpha}}+\frac{1}{8}\|D_x^{\frac{\alpha+1}{2}}u_m\|_{L^2}^2. \qquad (5.18)$$

Similarly, we derive that

$$\mathrm{E}\left(\sup_{t\in[0,T_0]}\|u_m(t)\|_{L^2}^{\frac{2(1+2\alpha)}{\alpha}}\right)\le C\mathrm{E}\left(\|u_{0m}\|_{L^2}^{\frac{2(1+2\alpha)}{\alpha}}\right)+CT\|\Phi_m\|_{L_2^{0,0}}^{C(\alpha)} \qquad (5.19)$$

and

$$\mathrm{E}\left(\sup_{t\in[0,T_0]}\|u_m(t)\|_{L^2}^4\right)\le C\mathrm{E}\left(\|u_{0m}\|_{L^2}^4\right)+CT\|\Phi_m\|_{L_2^{0,0}}^4, \qquad (5.20)$$

$$\mathrm{E}\left(\sup_{t\in[0,T_0]}\|u_m(t)\|_{L^2}^2\right)\le C\mathrm{E}\left(\|u_{0m}\|_{L^2}^2\right)+CT\|\Phi_m\|_{L_2^{0,0}}^2, \qquad (5.21)$$

where $C(\alpha)$ is a constant relative to α. Combining (5.18) with (5.19), we get

$$\mathrm{E}\left(\sup_{t\in[0,T]}\|u_m\|_{L^4}^4\right)\le C\mathrm{E}\left(\|u_{0m}\|_{L^2}^{\frac{2(1+2\alpha)}{\alpha}}\right)+CT\|\Phi_m\|_{L_2^{0,0}}^{C(\alpha)}+\frac{1}{8}\mathrm{E}\left(\sup_{t\in[0,T]}\|D_x^{\frac{\alpha+1}{2}}u_m\|_{L^2}^2\right). \qquad (5.22)$$

Combining (5.10), (5.13), (5.17), (5.20)-(5.22) with (5.8), we obtain

$$\begin{aligned}
&\frac{1}{2}\mathrm{E}\left(\sup_{t\in[0,T]}\int_{\mathbf{R}}\left[u_m^2+(D_x^{\frac{\alpha+1}{2}}u_m)^2\right]dx\right)\\
&=\frac{1}{2}\mathrm{E}\left(\sup_{t\in[0,T]}\int_{\mathbf{R}}u_m^2dx\right)+\mathrm{E}\left(\sup_{t\in[0,T]}I(u_m)\right)+\frac{1}{4}\mathrm{E}\int_{\mathbf{R}}u_m^4dx\\
&\le C\mathrm{E}\int_{\mathbf{R}}\left[u_{0m}^2+(D_x^{\frac{\alpha+1}{2}}u_{0m})^2\right]dx+C\mathrm{E}\left(\|u_{0m}\|_{L^2}^{\frac{2(2+3\alpha)}{\alpha}}\right)\\
&\quad+C\mathrm{E}\left(\|u_{m0}\|_{L^2}^{\frac{2(1+2\alpha)}{\alpha}}\right)+\frac{1}{4}\mathrm{E}\left(\sup_{t\in[0,T]}\int_{\mathbf{R}}\left[u_m^2+(D_x^{\frac{\alpha+1}{2}}u_m)^2\right]dx\right)\\
&\quad+C\mathrm{E}\left(\|u_{0m}\|_{L^2}^4\right)+CT\|\Phi_m\|_{L_2^{0,\frac{\alpha+1}{2}}}^{C(\alpha)}+CT\|\Phi_m\|_{L_2^{0,\frac{\alpha+1}{2}}}^{D(\alpha)}\\
&\quad+CT\|\Phi_m\|_{L_2^{0,\frac{\alpha+1}{2}}}^{E(\alpha)}+CT^2\|\Phi_m\|_{L_2^{0,\frac{\alpha+1}{2}}}^{4}+C,
\end{aligned}\tag{5.23}$$

where $C(\alpha),D(\alpha),E(\alpha)$ are constants relative to α.

From (5.22), by using the Young inequality, we have that

$$\begin{aligned}
&\mathrm{E}\left(\sup_{t\in[0,T]}\int_{\mathbf{R}}\left[u_m^2+(D_x^{\frac{\alpha+1}{2}}u_m)^2\right]dx\right)\\
&\le C\mathrm{E}\int_{\mathbf{R}}\left[u_{0m}^2+(D_x^{\frac{\alpha+1}{2}}u_{0m})^2\right]dx+C\mathrm{E}\left(\|u_{0m}\|_{L^2}^{\frac{2(2+3\alpha)}{\alpha}}\right)+C\mathrm{E}\left(\|u_{m0}\|_{L^2}^{\frac{2(1+2\alpha)}{\alpha}}\right)\\
&\quad+CT\|\Phi_m\|_{L_2^{0,\frac{\alpha+1}{2}}}^{C(\alpha)}+CT\|\Phi_m\|_{L_2^{0,\frac{\alpha+1}{2}}}^{D(\alpha)}+CT\|\Phi_m\|_{L_2^{0,\frac{\alpha+1}{2}}}^{E(\alpha)}+C+C\mathrm{E}\left(\|u_{0m}\|_{L^2}^4\right)\\
&\le C\mathrm{E}\int_{\mathbf{R}}\left[u_{0m}^2+(D_x^{\frac{\alpha+1}{2}}u_{0m})^2\right]dx+C\mathrm{E}\left(\|u_{0m}\|_{L^2}^{\frac{2(2+3\alpha)}{\alpha}}\right)\\
&\quad+CT\|\Phi_m\|_{L_2^{0,\frac{\alpha+1}{2}}}^{F(\alpha)}(1+T)+C+CT^2,
\end{aligned}\tag{5.24}$$

where $F(\alpha)=\max\{4,C(\alpha),D(\alpha),E(\alpha)\}$.

From $u_{0m}\longrightarrow u_0$ in $L^2(\Omega;L^\infty(0,T;H^{\frac{\alpha+1}{2}}(\mathbf{R}))\cap L^{\frac{2(2+3\alpha)}{\alpha}}(\Omega;L^2(\mathbf{R}))$ and $\Phi_m\longrightarrow\Phi$ in $L_2^{0,\frac{\alpha+1}{2}}$, we know that $\forall\epsilon>0$, there exists sufficiently large $m\in N^+$ such that

$$\begin{aligned}
&\mathrm{E}\|u_{0m}\|_{H^{\frac{\alpha+1}{2}}}^2+\mathrm{E}\left(\|u_{0m}\|_{L^2}^{\frac{2(2+3\alpha)}{\alpha}}\right)+\|\Phi_m\|_{L_2^{0,\frac{\alpha+1}{2}}}^{F(\alpha)}\\
&\le\mathrm{E}\|u_0\|_{H^{\frac{\alpha+1}{2}}}^2+\mathrm{E}\left(\|u_0\|_{L^2}^{\frac{2(2+3\alpha)}{\alpha}}\right)+\|\Phi\|_{L_2^{0,\frac{\alpha+1}{2}}}^{F(\alpha)}+C+CT^2+\epsilon.
\end{aligned}\tag{5.25}$$

Combining (5.24) with (5.25), we have that

$$\begin{aligned}
&\mathrm{E}\left(\sup_{t\in[0,T]}\int_{\mathbf{R}}\left[u_m^2+(D_x^{\frac{\alpha+1}{2}}u_m)^2\right]dx\right)\\
&\leq C\mathrm{E}\|u_0\|^2_{H^{\frac{\alpha+1}{2}}}+C\mathrm{E}\left(\|u_0\|_{L^2}^{\frac{2(2+3\alpha)}{\alpha}}\right)+CT\|\Phi\|^{F(\alpha)}_{L_2^{0,\frac{\alpha+1}{2}}}(1+T)+C\epsilon.
\end{aligned} \tag{5.26}$$

Combining (5.26) with (5.5), we have that

$$\begin{aligned}
&\mathrm{E}\left(\sup_{t\in[0,T]}\int_{\mathbf{R}}\left[v_m^2+(D_x^{\frac{\alpha+1}{2}}v_m)^2\right]dx\right)\\
&\leq C\left(\mathrm{E}\|u_0\|^2_{H^{\frac{\alpha+1}{2}}},\mathrm{E}\left(\|u_0\|_{L^2}^{\frac{2(2+3\alpha)}{\alpha}}\right),T,\|\Phi\|^{F(\alpha)}_{L_2^{0,\frac{\alpha+1}{2}}}\right).
\end{aligned} \tag{5.27}$$

Thus, combining the the local existence of solution with (5.28), we obtain that (5.3)-(5.4) possesses a unique global solution. Consequently, (5.1)-(5.2) possesses a unique global solution u_m P-a.s in $L^\infty(0,T;H^{\frac{\alpha+1}{2}}(\mathbf{R}))$ for any $T>0$.

The proof of Lemma 5.1 has been completed. □

Theorem 5.1. *Let $0<\alpha\leq 1$, $u_0\in L^2(\Omega;H^{\frac{\alpha+1}{2}}(\mathbf{R}))\cap L^{\frac{2(2+3\alpha)}{\alpha}}(\Omega;L^2(\mathbf{R}))$ be $\mathcal{F}_0$ -measurable. Then the Cauchy problem for (1.1) possesses a unique global solution $u\in L^2(\Omega;H^{\frac{\alpha+1}{2}}(\mathbf{R}))$ with $k=2$.*

Proof. From (5.25), we know that after extraction of a subsequence, we can find a function

$$\tilde{u}\in L^2(\Omega;L^\infty(0,T;H^{\frac{\alpha+1}{2}}(\mathbf{R})))$$

such that

$$u_m\rightharpoonup\tilde{u} \tag{5.28}$$

in $L^2(\Omega;L^\infty(0,T_0;H^{\frac{\alpha+1}{2}}(\mathbf{R})))$ weak star. Moreover, we have that

$$\mathrm{E}\left(\sup_{t\in[0,T]}\|\tilde{u}\|^2_{H^{\frac{\alpha+1}{2}}}\right)\leq C\mathrm{E}\left(\|u_0\|^2_{H^{\frac{\alpha+1}{2}}}\right)+CT\|\Phi\|^{F(\alpha)}_{L_2^{0,\frac{\alpha+1}{2}}}+CT+C. \tag{5.29}$$

Now we define the mapping

$$G_mv=U(t)u_{0m}+\int_0^tU(t-\tau)\left\{\frac{1}{3}\partial_x(v^3)\right\}d\tau+\overline{u}_m, \tag{5.30}$$

it is easily checked that G_m is a strict contraction uniformly on $B_{r_w}^{t_w}$, where

$$r_w\geq C\left[\left(\sup_{m\in\mathbf{N}}\|\Psi\overline{u}_m\|_{X_{\frac{\alpha+1}{2},b}}\right)+K_2\|\tilde{u}\|_{L^\infty(0,T;H^{\frac{\alpha+1}{2}})}\right]$$

and

$$2Ct_w^{1-2b}\left(r_w+K_2\|\tilde{u}\|_{L^\infty(0,T_0;H^{\frac{\alpha+1}{2}})}\right)^2\leq 1.$$

It is easily checked that G_m has a unique fixed point u_m for sufficiently large $m\in \mathrm{N}^+$. Now we prove that

$$u_m\to u$$

in $X^{t_w}_{\frac{\alpha+1}{2},b}$. Let $z_m=U(t)u_{0m}, v_m=u_m-z_m-\overline{u}_m$. Then

$$v_m(t)=-\frac{1}{2}\int_0^t U(t-s)\partial_x\left[(v_m+z_m(t)+\overline{u}_m)^3\right]ds. \tag{5.31}$$

Combining lemma 4.2 with (5.31), by using Lemmas 2.4-2.6, we have that

$$\begin{aligned}
&\|v_m-v\|_{X^{t_w}_{\frac{\alpha+1}{2},b}}\\
&\leq Ct_w^{1-2b}\left[\|v\|^2_{X^{t_w}_{\frac{\alpha+1}{2},b}}+\|v_m\|^2_{X^{t_w}_{\frac{\alpha+1}{2},b}}+\|z\|^2_{X^{t_w}_{\frac{\alpha+1}{2},b}}+\|z_m\|^2_{X^{t_w}_{\frac{\alpha+1}{2},b}}+\|\overline{u}\|^2_{X^{t_w}_{\frac{\alpha+1}{2},b}}+\|\overline{u}_m\|^2_{X^{t_w}_{\frac{\alpha+1}{2},b}}\right]\\
&\quad\times\left[\|v_m-v\|_{X^{t_w}_{\frac{\alpha+1}{2},b}}+\|z_m-z\|_{X^{t_w}_{\frac{\alpha+1}{2},b}}+\|\overline{u}-\overline{u}_m\|_{X^{t_w}_{\frac{\alpha+1}{2},b}}\right]\\
&\leq Ct_w^{1-2b}\left[\|u\|^2_{X^{t_w}_{\frac{\alpha+1}{2},b}}+\|u_m\|^2_{X^{t_w}_{\frac{\alpha+1}{2},b}}+\|z\|^2_{X^{t_w}_{\frac{\alpha+1}{2},b}}+\|z_m\|^2_{X^{t_w}_{\frac{\alpha+1}{2},b}}+\|\overline{u}\|^2_{X^{t_w}_{\frac{\alpha+1}{2},b}}+\|\overline{u}_m\|^2_{X^{t_w}_{\frac{\alpha+1}{2},b}}\right]\\
&\quad\times\left[\|v_m-v\|_{X^{t_w}_{\frac{\alpha+1}{2},b}}+\|z_m-z\|_{X^{t_w}_{\frac{\alpha+1}{2},b}}+\|\overline{u}-\overline{u}_m\|_{X^{t_w}_{\frac{\alpha+1}{2},b}}\right]\\
&\leq Ct_w^{1-2b}\left[\|u_0\|^2_{H^{\frac{\alpha+1}{2}}}+\|u_{0m}\|^2_{H^{\frac{\alpha+1}{2}}}+r_w\right]\\
&\quad\times\left[\|v_m-v\|_{X^{t_w}_{\frac{\alpha+1}{2},b}}+\|z_m-z\|_{X^{t_w}_{\frac{\alpha+1}{2},b}}+\|\overline{u}-\overline{u}_m\|_{X^{t_w}_{\frac{\alpha+1}{2},b}}\right]\\
&\leq\frac{1}{2}\left[\|v_m-v\|_{X^{t_w}_{\frac{\alpha+1}{2},b}}+\|z_m-z\|_{X^{t_w}_{1,b}}+\|\overline{u}-\overline{u}_m\|_{X^{t_w}_{\frac{\alpha+1}{2},b}}\right].
\end{aligned}\tag{5.32}$$

From (5.32), we have that

$$\|v_m-v\|_{X^{t_w}_{\frac{\alpha+1}{2},b}}\leq\|z_m-z\|_{X^{t_w}_{\frac{\alpha+1}{2},b}}+\|\overline{u}-\overline{u}_m\|_{X^{t_w}_{\frac{\alpha+1}{2},b}}\leq C\|u_{0m}-u_0\|_{H^{\frac{\alpha+1}{2}}}+\|\overline{u}-\overline{u}_m\|_{X^{t_w}_{\frac{\alpha+1}{2},b}} \tag{5.33}$$

From Lemma 2.6, we have that

$$E(\sup_{t\in[0,T]}\|\Psi\overline{u}-\Psi\overline{u}_m\|^2_{X_{\frac{\alpha+1}{2},b}})\leq M(b,\Psi)\|\Phi\|^2_{L_2^{0,\frac{\alpha+1}{2}}}. \tag{5.34}$$

Combining (5.33), (5.34) with the fact that

$$u_{0m}\longrightarrow u_0. \tag{5.35}$$

in $H^{\frac{\alpha+1}{2}}$ a.s., we have that

$$u_m \to u \tag{5.36}$$

in $X^{t_w}_{\frac{\alpha+1}{2},b}$. From (5.36), we have that

$$u = \widetilde{u} \tag{5.37}$$

on $[0, t_w]$ and

$$\|u(t_w)\|_{H^{\frac{\alpha+1}{2}}} \leq \|\widetilde{u}\|_{L^\infty(0,T;H^{\frac{\alpha+1}{2}})}. \tag{5.38}$$

Combining (5.38) with Theorem 4.1, we can construct a solution on $[t_w, 2t_w]$; starting from $u(2t_w)$, we obtain a solution on $[0, T]$ by reiterating this argument. Thus, the proof of Theorem 5.1 has been completed. $\square$

Acknowledgment

We would like to thank an anonymous referee for the valuable suggestion for improving the quality of this paper.

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